THE STABILITY OF THE MOTION OF A LIQUID, DUE TO THERMOCAPILLARY FORCES

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A study is made of the stability of the plane-parallel flow of a viscous liquid in a layer with a free boundary, under weightless conditions. The motion of the liquid is due to the dependence of the surface tension on the temperature. An exact solution for an unperturbed boundary is obtained by the same method used in [1], but with a more general boundary condition for the temperature. A study of the stability was carried out by the method of small vibrations, taking account of the perturbation of the free boundary. The article discusses the asymptotic behavior of long waves at small Reynolds numbers, and the conditions for instability are found.

1. Statement of Problem. Under weightless conditions, we consider a layer of viscous incompressible liquid, bounded on the one hand by a free surface, and on the other hand by a solid wall (Fig. 1). We postulate that a surface tension with the coefficient σ, depending linearly on the temperature T, is acting at the free boundary. Let a constant temperature gradient be given along the solid wall. At the free boundary we assume that the heat flux through the surface is proportional to the difference in temperature between the liquid medium and the external medium, whose temperature is determined in accordance with a linear law, with the same gradient, A, as in the solid wall. We shall assume that, at the surface of the liquid, a given pressure, whose dimensionless value, p_1, is shown below, is acting from the side of the external medium.

As units of length, time, mass, and temperature, respectively, we take the quantities l, $\rho\nu\,(-A d\sigma/dT)^{-1}$, ρl^3, Al, where l is the mean thickness of the layer of liquid, ρ is the density, and ν is the coefficient of kinematic viscosity.

The equations of motion and the boundary conditions, in dimensionless variables, have the form

$$R\,[\mathbf{v}_t + (\mathbf{v}, \nabla)\,\mathbf{v}] = \Delta\mathbf{v} - \nabla p \tag{1.1}$$

$$RP\,(T_t + \mathbf{v}\nabla T) = \Delta T, \qquad \operatorname{div}\mathbf{v} = 0 \tag{1.2}$$

$$p_{nn} + p_1 = RW^{-1}N_{xx}(1 + N_x{}^2)^{-3/2} \qquad (y = 1 + N) \tag{1.3}$$

$$p_{ns} = -\,\partial T / \partial s \qquad (y = 1 + N) \tag{1.4}$$

$$N_t + v_x N_x = v_y \qquad (y = 1 + N) \tag{1.5}$$

$$\partial T / \partial n + m\,(T - T_e) = 0 \qquad (y = 1 + N) \tag{1.6}$$

$$T = T_e, \qquad \mathbf{v} = 0 \qquad (y = 0) \tag{1.7}$$

where $\mathbf{v}$ is the velocity of the liquid; p is the pressure; n is the external normal to the liquid; s is the tangent; $N = N(x, t)$ is the perturbation of the free boundary; $p_1 = -1.5x + \text{const}$; $T_e = x + \text{const}$ is the temperature of the external medium.

The problem contains four dimensionless parameters: $R = Vl\nu^{-1}$ is the Reynolds number; $W = \rho V^2 l \sigma^{-1}$ is the Weber number; $P = \nu/\chi$ is the Prandtl number (χ is the coefficient of thermal diffusivity); m is the heat-transfer coefficient; and $V = -Al(\rho\nu)^{-1} d\sigma/dT$ is the characteristic velocity.

It can be verified that the formulas

$$\begin{gathered} v_{x0} = u = -\,{}^3/_4 y^2 + {}^1/_2 y, \qquad v_{y0} = 0, \qquad p_0 = p_1 \\ T_0 = x + RP\left(-\frac{1}{16}y^4 + \frac{1}{12}y^3 - \frac{1}{48}\frac{m}{1+m}y\right) + \text{const} \end{gathered} \tag{1.8}$$

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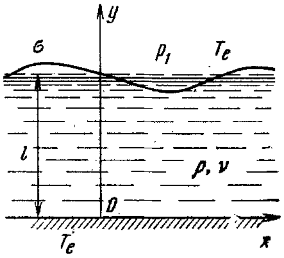

Fig. 1

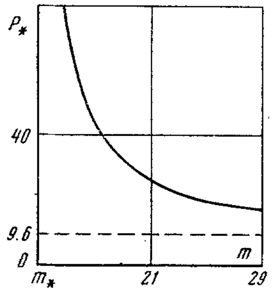

Fig. 2

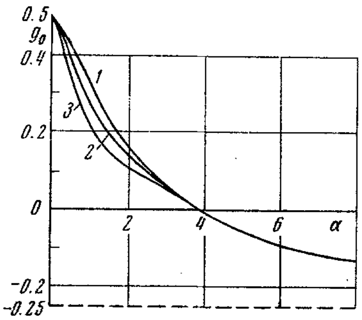

Fig. 3

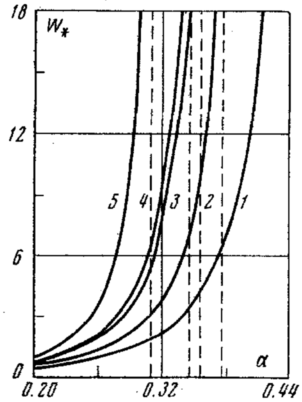

Fig. 4

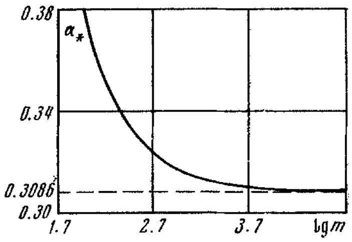

Fig. 5

yield a steady-state solution to the problem (1.1)-(1.7) for the unperturbed free boundary (N = 0) under the conditions of open flow [1]. In what follows, we shall investigate the stability of this solution.

Linearizing the system (1.1)-(1.7) in the neighborhood of the solution (1.8), eliminating the pressure, and introducing a flow function using the relationships

$$v_x = v_{x0} + \psi_y, \quad v_y = -\psi_x \tag{1.9}$$

we arrive at the problem

$$\Delta^2\psi = R(\Delta\psi_t + u\Delta\psi_x - u''\psi_x) \tag{1.10}$$

$$\Delta T = RP(T_t + uT_x + \psi_y - T_{0y}\psi_x) \tag{1.11}$$

$$R(\psi_{yt} - {}^1/_4\psi_{xy} + \psi_x) - \Delta\psi_y - 2\psi_{xxy} + 2N_{xx} - RW^{-1}N_{xxx} = 0 \quad (y = 1) \tag{1.12}$$

$$\psi_{yy} - \psi_{xx} + u''N + T_x = 0 \quad (y = 1) \tag{1.13}$$

$$N_t + uN_x + \psi_x = 0 \quad (y = 1) \tag{1.14}$$

$$T_y + T_{0yy}N - T_{0x}N_x + m(T + T_{0y}N) = 0 \quad (y = 1) \tag{1.15}$$

$$T = 0, \quad \psi_x = \psi_y = 0 \quad (y = 0) \tag{1.16}$$

We then separate the time and the longitudinal coordinate x, setting

$$(\psi, T, N) = (\varphi, \theta, a)\, e^{i\alpha(x-ct)} \tag{1.17}$$

where $\varphi = \varphi(y)$; $\theta = \theta(y)$; a is a constant; α is the wave number; $c = c_r + ic_i$ is the complex frequency.

We draw a conclusion as to the stability as a function of the sign of the imaginary part of the quantity c: if, for all eigenvalues $c_i < 0$, solution (1.8) is stable; if there is even a single eigenvalue for which $c_i > 0$, we have instability; the case $c_i = 0$ corresponds to the limit of stability (neutral perturbations).

For normalization, we select the condition

$$a = 1 \tag{1.18}$$

Taking account of (1.8), (1.17), and (1.18), problem (1.10)-(1.16), is transformed to the form

$$\psi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha R\,[(u-c)(\varphi'' - \alpha^2\varphi) - u''\varphi] \tag{1.19}$$

$$\theta'' - \alpha^2\theta = RP\,\{i\alpha\,[(u-c)\,\theta - T_{0y}\varphi] + \varphi'\} \tag{1.20}$$

$$\varphi''' + [i\alpha R\,(c + {}^1/_4) - 3\alpha^2]\,\varphi' - i\alpha R\varphi + 2\alpha^2 - i\alpha^3 RW^{-1} = 0 \qquad (y=1) \tag{1.21}$$

$$\varphi'' + \alpha^2\varphi + i\alpha\theta - 1.5 = 0 \qquad (y=1) \tag{1.22}$$

$$\theta' - i\alpha + m\left[\theta - \frac{mRP}{48\,(1+m)}\right] - \frac{1}{4}RP = 0 \qquad (y=1) \tag{1.23}$$

$$\theta = \varphi = \varphi' = 0 \qquad (y=0) \tag{1.24}$$

$$c = \varphi\,(1) - 0.25 \tag{1.25}$$

2. Asymptotic Behavior of Long Waves. Setting $\alpha = 0$ in (1.19)-(1.25), we obtain

$$\varphi_0'''' = 0, \qquad \theta_0'' = RP\varphi_0' \tag{2.1}$$

$$\varphi_0''' = 0, \qquad \varphi_0'' = 1.5 \qquad (y=1) \tag{2.2}$$

$$\theta_0' + m\theta_0 = RP\left[\frac{m^2}{48\,(1+m)} + \frac{1}{4}\right] \qquad (y=1) \tag{2.3}$$

$$\theta_0 = \varphi_0 = \varphi_0' = 0 \qquad (y=0) \tag{2.4}$$

$$c_0 = \varphi_0\,(1) - 0.25 \tag{2.5}$$

from which we find

$$\varphi_0 = {}^3/_4 y^2, \qquad \theta_0 = RPy\,({}^1/_4 y^2 + 5P_*^{-1}) \tag{2.6}$$

$$c_0 = 0.5 \tag{2.7}$$

where

$$P_* = 9.6\,\frac{(m+1)^2}{(m-6)^2 - 48} \tag{2.8}$$

Assuming that, at small values of α, problem (2.1)-(2.5) is a generatrix for the system (1.19)-(1.25), we seek the solution of the latter in the form

$$\varphi = \sum_{k=0}^{\infty} (i\alpha)^k \varphi_k, \qquad \theta = \sum_{k=0}^{\infty} (i\alpha)^k \theta_k, \qquad c = \sum_{k=0}^{\infty} (i\alpha)^k c_k \tag{2.9}$$

At sufficiently small values of the wave numbers α, the sign of the imaginary part of c_i, by virtue of the equality

$$c_i = \alpha c_1 + O\,(\alpha^3)$$

is determined by the sign of the coefficient c_1, which is calculated with solution of the problem

$$\varphi_1'''' = R\,[(u - {}^1/_2)\,\varphi_0'' - u''\varphi_0] \tag{2.10}$$

$$\theta_1'' = RP\,[(u - {}^1/_2)\,\theta_0' - T_{0y}\varphi_0 + \varphi_1'] \tag{2.11}$$

$$\varphi_1''' = R\,(\varphi_0 - {}^3/_4\varphi_0'), \qquad \varphi_1'' = -\theta_0 \qquad (y=1) \tag{2.12}$$

$$\theta_1' + m\theta_1 = 1 \qquad (y=1) \tag{2.13}$$

$$\theta_1 = \varphi_1 = \varphi_1' = 0 \qquad (y=0) \tag{2.14}$$

$$c_1 = \varphi_1\,(1) \tag{2.15}$$

Solving system (2.10)-(2.15), for c_1 we obtain the value

$$c_1 = {}^1/_{10} R\,(1 - P/P_*) \tag{2.16}$$

Analysis of formula (2.16) shows that, at $0 \le m \le m_* = 6 + 4\sqrt{3}$, we have long-wave instability for any values of the P number. If $m > m_*$, the result depends on the value of the Prandtl number: at $P > P_*$ we have stability, while for $P < P_*$ we have instability. Figure 2 shows a curve of the dependence $P_* = P_*(m)$. We note that at $P < 9.6$ the solution of (1.8) is unstable for any values of the heat-transfer coefficient m.

3. The Case of Small Reynolds Numbers R and Arbitrary Wave Numbers α. Assuming smallness of the Reynolds number, we have the solution of the problem (1.19)-(1.25) in the form of series (3.1):

$$\varphi = \sum_{k=0}^{\infty} R^k \psi_k, \qquad \theta = \sum_{k=0}^{\infty} R^k \tau_k, \qquad c = \sum_{k=0}^{\infty} R^k g_k \tag{3.1}$$

We substitute expansions (3.1) into Eqs. (1.19)-(1.25), and collect the terms with identical powers of the small parameter R; with R° we find

$$\psi_0'''' - 2\alpha^2\psi_0'' + \alpha^4\psi_0 = 0, \quad \tau_0'' - \alpha^2\tau_0 = 0 \tag{3.2}$$

$$\psi_0''' - 3\alpha^2\psi_0' + 2\alpha^2 = 0 \quad (y = 1) \tag{3.3}$$

$$\psi_0'' + \alpha^2\psi_0 + i\alpha\tau_0 - 1.5 = 0 \quad (y = 1) \tag{3.4}$$

$$\tau_0' + m\tau_0 - i\alpha = 0 \quad (y = 1) \tag{3.5}$$

$$\psi_0 = \psi_0' = \tau_0 = 0, \quad (y = 0) \tag{3.6}$$

$$g_0 = \psi_0(1) - 0.25 \tag{3.7}$$

Calculations give the following value for the coefficient g_0:

$$g_0 = h_1(\mathrm{ch}\alpha - \alpha^{-1}\mathrm{sh}\alpha) + h_2\mathrm{sh}\alpha - 0.25 \tag{3.8}$$

where

$$h_1 = \frac{\alpha^2\,\mathrm{sh}\,\alpha}{2(\alpha + m\,\mathrm{th}\,\alpha)(\alpha^2 + \mathrm{ch}^2\alpha)} - \frac{\alpha\,\mathrm{sh}\,\alpha}{\alpha^2 + \mathrm{ch}^2\alpha} - \frac{\mathrm{ch}\,\alpha}{4(\alpha^2 + \mathrm{ch}^2\alpha)} \tag{3.9}$$

$$h_2 = (\alpha^{-1} - \mathrm{th}\alpha)h_1 + (\alpha\mathrm{ch}\alpha)^{-1} \tag{3.10}$$

Figure 3 gives a plot of the dependence of the main part g_0 of the phase velocity c on the wave number α for different values of the coefficient m. Curves 1, 2, and 3 correspond to $m=0$, 1, and $5 \cdot 10^6$. With an arbitrary value of m, in the interval from 0 to $5 \cdot 10^6$, the corresponding curves for $g_0 = g_0(\alpha)$ are included between curves 1 and 3. Short-wave perturbations are localized near the free surface, while their rate of propagation, as calculations show, differs only slightly from 0.25, i.e., the value of the velocity of the main flow $v_x = u$ at the boundary. The coefficient g_1 is determined with solution of the problem

$$\psi_1'''' - 2\alpha^2\psi_1'' + \alpha^4\psi_1 = i\alpha[(u - g_0)(\psi_0'' - \alpha^2\psi_0) - u''\psi_0] \tag{3.11}$$

$$\tau_1'' - \alpha^2\tau_1 = P[i\alpha(u - g_0)\tau_0 + \psi_0'] \tag{3.12}$$

$$\psi_1''' - 3\alpha^2\psi_1' = i\alpha[\psi_0 - (g_0 + 1/4)\psi_0' + \alpha^2 / W] \quad (y = 1) \tag{3.13}$$

$$\psi_1'' + \alpha^2\psi_1 = -i\alpha\tau_1 \quad (y = 1) \tag{3.14}$$

$$\tau_1' + m\tau_1 = P\left[\frac{m^2}{48(1+m)} + \frac{1}{4}\right] \quad (y = 1) \tag{3.15}$$

$$\psi_1 = \psi_1' = \tau_1 = 0 \quad (y = 0) \tag{3.16}$$

and is represented by the formula

$$g_1 = ik_2(1 - W_*/W) \tag{3.17}$$

where $W_* = k_1k_2^{-1}$, $k_1 = k_1(\alpha) > 0$, $k_2 = k_2(\alpha, m, P)$ are known real functions, whose explicit expressions are not given here in view of their cumbersome nature.

We have

$$\mathrm{Im}\, c = k_2(1 - W_*W^{-1})R + o(R) \tag{3.18}$$

Consequently, with small Reynolds numbers, the conclusion with regard to the stability depends on the sign of the quantity $k_2(1 - W_*W^{-1})$.

A numerical analysis was made for a Prandtl number equal to 7.3 (water). An investigation was made of the dependence of the critical value of the Weber number W_* on the wave number at different values of m. Figure 4 gives curves 1, 2, 3, 4, 5, corresponding to values of the heat-transfer coefficient $m = 50$, 100, 150, 200, $5 \cdot 10^3$, and $5 \cdot 10^6$.

At $\alpha = 0$, the value of W_* reverts to zero; at $\alpha \to \alpha_*$, W_* rises infinitely. At the point α_*, the coefficient k_2 changes sign: $k_2 > 0$ for $\alpha < \alpha_*$, and $k_2 < 0$ for $\alpha > \alpha_*$. Figure 5 gives a curve of the dependence of α_* on log m.

The results of calculations ($P = 7.3$) lead to the following conclusions: 1) perturbations with wave numbers $\alpha > \alpha_*$ are damped with time; 2) perturbations with wave numbers $0 < \alpha < \alpha_*$ behave differently, depending on the Weber number W: at $W < W_*$ they are damped; at $W > W_*$ they increase.

We note in conclusion that the convergence of the series (2.9) and (3.1) can be demonstrated by reducing the problem to an integral Fredholm equation of the second order, and then applying the principle of compressive reflections and the theorem of an implicit function [2].

LITERATURE CITED

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